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Accelerated Projection Methods

MORTON B. FRIEDMAN

Department of Civil Engineering and Engineering Mechanics, Columbia University, New York, New York 10027

AND

DAVID R. DELLWO

Department of Mathematics and Science, United States Merchant Marine Academy, Kings Point, New York 11024

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A new approach to the theory of projection methods is developed in which the objective is to determine a projected solution to any desired accuracy for a fixed choice of the projection. Neither the structure nor the range of the projection is altered to achieve increased accuracy. Instead, an operator equation is derived for the projected solution. Successive approximations of this equation give rise to a sequence of accelerated projection methods whose solutions converge to the projected solution. Error expressions are obtained and several numerical examples of solving linear integral equations of the second kind are given.

1.0 INTRODUCTION

Projection methods form a class of well-known numerical techniques for constructing approximate solutions of the operator equation $y = g + Ky$. Here y and g are elements of a Banach space and the compact linear operator K is such that $I - K$ is invertible. These methods employ a bounded projection P to represent the solution in the form $y = Py + (I - P)y$. If P is chosen appropriately, Py provides a good approximation to y. But in general, Py can be determined only approximately.

P is often chosen so that its range is finite-dimensional and an approximation to Py is obtained by solving a finite system of linear algebraic equations. Such is the case, for example, with the well-known methods of collocation [1–3], moments [4–6], and Galerkin [7–10]. They provide approximate solutions $\tilde{y}$ satisfying

$$\|Py - \tilde{y}\| = \mathcal{O}(\max(\|g - Pg\|, \|K - PK\|)),$$

where $\|\cdot\|$ denotes the appropriate norm. If the accuracy obtained with a given choice of P is inadequate it is necessary to consider a projection P with a more complicated structure and/or a range of larger dimension so that $\|g - Pg\|$ and

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$\|K - PK\|$ can be made smaller. This results in computational complexities associated with increasingly large systems of equations.

The present work develops a different approach to projection methods in which the objective is to determine Py to any desired accuracy for a fixed choice of P. Neither the structure of P nor its range are altered to achieve increased accuracy. Instead, an operator equation is derived for Py. This equation is developed in the form of an infinite series defined by the iterates of $K - PK$. Successive truncations of the series defines a sequence of higher order schemes whose solutions provide a sequence of approximations to Py. Most importantly, the equations to be solved in the higher order schemes are no more complex than that for the first order approximation. Only the routine computations to determine the elements in the scheme increase in amount with the order. An analysis of the nth truncation provides the estimate

$$\|Py - u_n\| = \mathcal{O}(\|K - PK\|^n)$$

for the accuracy of the nth approximation u_n to Py. The first order scheme $n = 1$, yields an approximation to Py which is essentially equivalent to that obtained from the standard methods since the latter are typically in error by $\mathcal{O}(\|K - PK\|)$. The second order scheme $n = 2$, results in an error $\mathcal{O}(\|K - PK\|^2)$. To achieve a comparable accuracy with the first order scheme requires in general, a significant increase in the complexity of P and in the corresponding computational effort.

This can be seen readily in the particular case of the projections used to generate a collocation scheme. Here the range of P is finite dimensional and $Py = P_N y$ depends on the values of y at N preassigned mesh points. Each of the approximates u_n is obtained by solving an $N \times N$ system of linear equtions whose solution approximates y at the mesh points. Typically $\|K - P_N K\| = \mathcal{O}(1/N)$. Hence doubling the number of nodes by changing P_N to P_{2N} will only halve the error in the first order scheme at the $2N$ nodes. But the second order scheme associated with P_N will produce a far smaller error $\mathcal{O}(1/N^2)$ at the N nodes. Thus the nodal values of y can be obtained to any desired accuracy without increasing the number of nodes. Accelerated convergence at selected mesh points has been discussed previously, see [11]. However, these superconvergence results apply only at "Gaussian" mesh points and moreover cannot be improved without changing the projection. The important and distinctive features in the present technique are the arbitrariness of the mesh and the use of a fixed projection.

The primary objective generally is to obtain an accurate approximtion at the node points. But this does not ensure accurate interpolation between the nodes. A second operator equation for the solution y in terms of the projection Py is developed for this purpose. It is shown that each approximation to Py provides a corresponding approximation to y with an error of the same order as that associated with the nodes.

In the application to integral equations, it is necessary to evaluate one or more iterates of the kernel of K depending upon the order of approximation desired. For some kernels it may be possible and feasible to obtain several iterates by elementary integration. In a particular case, for some order schemes it may be necessary to resort

to numerical integration. An integration technique accurate to the order of the error in the scheme must be employed and this represents the most serious limitation to achieving an arbitrary accuracy.

The utility of this method and its superiority over existing techniques is demonstrated by treating several nonhomogeneous integral equations of the Fredholm type that have been studied previously. The first is a problem that can be solved by elementary means but provides a transparent model for exhibiting accelerated convergence even in the neighborhood of an eigenvalue. Two other examples treated are nontrivial problems drawn from different areas of applied physics. The first of these is the singular Kirkwood–Riseman equation that occurs in polymer physics. The second is Love's equation, which appears in electrostatics.

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2.1. *General Theory*

Consider the operator equation

$$y = Ky + g, \tag{2.1a}$$

where y and g are elements of a Banach space B, and K is a compact linear operator on B such that $I - K$ is invertible. Let P be a bounded linear projection on B. The object is to derive an equation for the projection of the solution to Eq. (2.1a). This is achieved as follows.[1]

Equation (2.1a) can be written

$$y = g + KPy + KQy, \tag{2.1b}$$

where $Q = I - P$. P applied to Eq. (2.1b) yields

$$Py = Pg + PKPy + PKQy \tag{2.2}$$

and Q applied to Eq. (2.1b) yields

$$Qy = Qg + QKPy + QKQy. \tag{2.3}$$

If $I - QK$ is invertible an equation for Py is obtained by solving (2.3) for Qy and then substituting into (2.2). This gives

$$Py - PK(I - QK)^{-1} Py = Pg + PK(I - QK)^{-1} Qg, \tag{2.4}$$

[1] The derivation of Eq. (2.4) presented above was suggested by one of the reviewers, I. H. Sloan. For the present purposes it is more transparent than the one originally provided by the authors.

which has Py as its only solution, since the operator

$$R = I - PK(I - QK)^{-1} = (I - K)(I - QK)^{-1}$$

is invertible. If $\|QK\| < 1$ the nth order projection scheme

$$u_n - PK \sum_{j=0}^{n-1} (QK)^j u_n = Pg + PK \sum_{j=0}^{n-1} (QK)^j Qg \tag{2.5}$$

is obtained from (2.4) by replacing $(I - QK)^{-1}$ with the $(n-1)$st truncation, $\sum_{j=0}^{n-1} (QK)^j$, of $(I - QK)^{-1} = \sum_{j=0}^{\infty} (QK)^j$.

The operator

$$R_n = I - PK \sum_{j=0}^{n-1} (QK)^j = (I - K + PK(QK)^n)(I - QK)^{-1}$$

is invertible if

$$\varepsilon_n = \|(I - K)^{-1} PK(QK)^n\| < 1 \tag{2.6}$$

with $\|R_n^{-1}\| \leqslant 2(1 - \varepsilon_n)^{-1} \|(I - K)^{-1}\|$. And it follows from (2.4) and (2.5) that

$$Py - u_n = R_n^{-1} PK(QK)^n (I - QK)^{-1} (Qg + Py) = R_n^{-1} PK(QK)^n y$$

so that the error associated with the nth order scheme is

$$\|Py - u_n\| = \mathcal{O}(\|K - PK\|^n). \tag{2.7}$$

Since $\{\varepsilon_n\}_{n=1}^{\infty}$ is a monotonic sequence decreasing to zero, it follows that (2.6) holds for all n if $\varepsilon_1 < 1$ and can always be achieved for sufficiently large n. For small values of n, condition (2.6) may be overly restrictive. Then the alternative conditions

$$\|T_n\| < 1 \tag{2.8}$$

obtained from $R_n = (I - K)(I + T_n)$ with

$$T_n = (I - K)^{-1} \left\{ QK - PK \sum_{j=1}^{n-1} (QK)^j \right\}$$

may be more useful.

The first and second order schemes are

$$u_1 - PKu_1 = Pg + PKQg, \tag{2.9}$$

$$u_2 - PKu_2 - PKQKu_2 = Pg + PKQg + PKQKQg. \tag{2.10}$$

Galerkin's method results from (2.9) by neglecting the term $PKQg$. Condition (2.8)

for the first order scheme is the same as that derived in [7] for Galerkin's method, namely,

$$\|(I-K)^{-1}QK\| < 1. \tag{2.11}$$

For the second order scheme, (2.8) is

$$\|(I-K)^{-1}(I-PK)QK\| < 1. \tag{2.12}$$

A representation for the solution y in terms of Py and g can be obtained from (2.1a) by applying Q to (2.1a) to yield

$$(I-QK)y = Py + Qg. \tag{2.13}$$

Then the error estimate

$$\|y - y_n\| = \mathcal{O}(\|K-PK\|^n)$$

for the approximate solution

$$y_n = \sum_{j=0}^{n-1} (QK)^j (u_n + Qg) \tag{2.14}$$

follows from (2.13) and (2.7). These approximate solutions represent iterates of the projection approximates u_n and thus generalize the standard approximate $y \approx y_1 = u_1 + Qg$.

2.2. *An Alternate Theory*

The discussion presented in the previous section is based on the assumption that $\|QK\| < 1$, a condition expressing the attempt to employ PK as an approximation to K. An alternate approach might employ KP; then it is necessary to require $\|KQ\| < 1$. In this case the analogue of (2.4)

$$Py - P(I-KQ)^{-1}KPy = P(I-KQ)^{-1}g$$

is obtained from (2.4) using the identity

$$K(I-QK)^{-1} = (I-KQ)^{-1}K.$$

And the nth order projection scheme

$$w_n - P\sum_{j=0}^{n-1}(KQ)^j Kw_n = P\sum_{j=0}^{n-1}(KQ)^j g \tag{2.15}$$

approximates Py in the sense

$$\|Py - w_n\| = \mathcal{O}(\|K-KP\|^n) \tag{2.16}$$

provided (2.6) is satisfied. Since (2.15) differs from (2.5) only in the inhomogeneous term, the first order scheme is (2.9) without the term $PKQg$ and is therefore the Galerkin method. The second order scheme is (2.10) without the term $P(KQ)^2 g$. In addition,

$$Py - w_n = R_n^{-1} P(KQ)^n y = R_n^{-1} P(KQ)^n Qy$$

and the estimate (2.16) is easily improved to include an extra factor of $\|Qy\|$:

$$\|Py - w_n\| = \mathcal{O}(\|K - KP\|^n \|Qy\|). \tag{2.17}$$

For $n = 1$, this coincides with a result given in [11] where first order methods alone are employed.

The approximate solution in this case is

$$y^{(n)} = \sum_{j=0}^{n-1} (KQ)^j (g + Kw_n). \tag{2.18}$$

It is obtained from (2.1a) written in the form

$$(I - KQ) y = g + KPy$$

and represents iterates of the projection approximates w_n. For $n = 1$, (2.18) coincides with results given in [3] for improving the Galerkin approximate. The error associated with (2.18) is of the same order as (2.17).

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3.1. *Applications*

In this section higher order projection schemes are used to analyze several examples of linear integral equations of the type

$$y(x) = g(x) + \lambda \int_a^b k(x, t)\, y(t)\, dt, \qquad a \leqslant x \leqslant b. \tag{3.1}$$

Here λ is a parameter, g is a bounded function and the integral operator

$$Ky = \lambda \int_a^b k(x, t)\, y(t)\, dt$$

is assumed to be a compact operator from $L^\infty[a, b]$ to $C[a, b]$, with $\|\cdot\|$ denoting the sup norm. The compactness of K is assured if $\|\int_a^b |k(x, t)|\, dt\|$ is finite and if

$$\theta(\varepsilon) = \sup_x \sup_{|x-t| \leqslant \varepsilon} \int_a^b |k(x, \xi) - k(t, \xi)|\, d\xi \to 0 \tag{3.2}$$

as $\varepsilon \downarrow 0$, see [7].

The projection methods employ a member P_δ, $0 < \delta \leqslant \delta_0$, of a one parameter family of bounded projections on $L^\infty[a, b]$ with a range R_δ satisfying

$$\inf_{h \in R_\delta} \|h - \eta\| = \mathcal{O}(m(\eta; \delta)) \qquad \text{as} \quad \delta \to 0$$

for each η in $C[a, b]$. Here $m(\eta, \delta)$ denotes the modulus of continuity of η. It follows that

$$\|K - P_\delta K\| = \mathcal{O}(|\lambda| \, \|P_\delta\| \, \theta(\delta)), \tag{3.3}$$

(see [2]) and thus the condition $\|QK\| < 1$ is satisfied if $\|P_\delta\| \, \theta(\delta) \to 0$ with $\delta \to 0$. For the examples presented here, $\|P_\delta\| = 1$ and therefore $\|QK\| < 1$ follows from (3.2) and (3.3).

3.2. *A Model Problem*

The degenerate Fredholm integral equation

$$y(x) = 1 + \lambda \int_0^1 e^{x-t} y(t) \, dt = 1 + Ky \tag{3.4}$$

is easily solved and provides insight into the method of Section (2.1). In addition comparisons with previously published results are possible. In [11], Eq. (3.4), with $\lambda = 2$, is used to demonstrate the superconvergence properties of piecewise polynomial Galerkin approximations, in particular a Galerkin method with a linear spline subspace. Consequently the projection used here is the piecewise linear interpolant

$$Py = \sum_{j=0}^{N} y(x_j) \, r_j(x), \qquad N \geqslant 1,$$

defined on a uniform mesh $0 = x_0 < x_1 < \cdots < x_N = 1$, $x_j = j/N$, with

$$\begin{aligned} r_j(x) &= \frac{x - x_{j-1}}{x_j - x_{j-1}}, && x_{j-1} < x \leqslant x_j, \\ &= \frac{x_{j+1} - x}{x_{j+1} - x_j}, && x_j \leqslant x < x_{j+1}, \\ &= 0, && \text{otherwise}, \end{aligned} \tag{3.5}$$

for $0 < j < N$ and

$$\begin{aligned} r_0(x) &= \frac{x_1 - x}{x_1 - x_0}, && x_0 \leqslant x < x_1, \\ &= 0, && \text{otherwise}. \end{aligned} \tag{3.6}$$

$$\begin{aligned} r_N(x) &= \frac{x - x_{N-1}}{x_N - x_{N-1}}, && x_{N-1} < x \leqslant x_N, \\ &= 0, && \text{otherwise} \end{aligned} \tag{3.7}$$

It follows from (2.4) that the interpolant $\bar{y} = Py$ of the solution to (3.4) satisfies

$$\bar{y}(x) - \lambda(1-\omega\lambda)^{-1}\left(\sum_{j=0}^{N} e^{x_j} r_j(x)\right)\int_0^1 e^{-t}\bar{y}(t)\,dt = 1 \tag{3.8}$$

provided N is selected so that $|\omega\lambda| < 1$. Here

$$\omega = \int_0^1 e^{-t}\left(e^t - \sum_{j=0}^{N} e^{x_j} r_j(t)\right) dt = \mathcal{O}(1/N^2). \tag{3.9}$$

For $\lambda \neq 1$ it is easy verified that the solution of (3.4) is

$$y(x) = 1 + \lambda(1-\lambda)^{-1}(1-e^{-1})\,e^x$$

and that

$$Py(x) = 1 + \lambda(1-\lambda)^{-1}(1-e^{-1})\sum_{j=0}^{N} e^{x_j} r_j(x) \tag{3.10}$$

is the solution of (3.8). For $\lambda = 1$ neither Eq. (3.4) nor (3.8) has a solution.

The nth order projection scheme (2.5) is

$$u_n(x) - \lambda(1-\omega\lambda)^{-1}(1-\omega^n\lambda^n)\left(\sum_{j=0}^{N} e^{x_j} r_j(x)\right)\int_0^1 e^{-t}u_n(t)\,dt = 1$$

and the nth order approximate to $\bar{y}$ is

$$u_n(x) = 1 + \lambda(1-e^{-1})(1-\omega^n\lambda^n)\,\Delta_n^{-1}(\lambda)\sum_{j=0}^{N} e^{x_j} r_j(x) \tag{3.11}$$

provided $\Delta_n(\lambda) \equiv 1 - \lambda + (1-\omega)\,\omega^n\lambda^{n+1} \neq 0$.

Equation (3.10) gives the interpolant $\bar{y}$ exactly and thus the error associated with (3.11) is

$$E_n(\lambda, N) = \max_x |\bar{y}(x) - u_n(x)| = |\lambda(1-\lambda)^{-1}(e-1)\,\omega^n\lambda^n(1-\omega\lambda)\,\Delta_n^{-1}(\lambda)|.$$

This expression is tabulated in Tables I.a, I.c and I.d for selected values of $\lambda = 2$, 1.001 and 1.00001, respectively. Table I.b contains results reported in [11] for the maximum error at the $2N$ Gauss points

$$\bar{x}_i = i/N + (1/2N)(1 \pm 1/\sqrt{3}), \qquad i = 0, 1, \ldots, N-1,$$

obtained from a linear spline Galerkin analysis of (3.4) with $\lambda = 2$.

An inspection of Table I.a and I.b shows that the second order scheme provides a accuracy at the uniform mesh points that is comparable with that achieved at the Gaussian points in [11]. The third and fourth order schemes each achieve an accuracy that is in excess of these superconvergence results.

TABLE I.a

Maximum Error Associated with (3.11) for $\lambda = 2$

N	1st order	2nd order	3rd order	4th order
2	0.1386 $E+00$	0.6344 $E-02$	0.2656 $E-03$	0.1116 $E-04$
4	0.3550 $E-01$	0.3784 $E-03$	0.3949 $E-05$	0.4123 $E-07$
8	0.8931 $E-02$	0.2339 $E-04$	0.6095 $E-07$	0.1588 $E-09$
16	0.2236 $E-02$	0.1458 $E-05$	0.9493 $E-09$	0.6181 $E-12$
32	0.5593 $E-03$	0.9106 $E-07$	0.1482 $E-10$	0.2412 $E-14$
64	0.1398 $E-03$	0.5690 $E-08$	0.2315 $E-12$	0.9421 $E-17$
128	0.3496 $E-04$	0.3556 $E-09$	0.3618 $E-14$	0.3680 $E-19$
256	0.8740 $E-05$	0.2223 $E-10$	0.5652 $E-16$	0.1437 $E-21$
512	0.2185 $E-05$	0.1389 $E-11$	0.8830 $E-18$	0.5614 $E-24$
1024	0.5457 $E-06$	0.8666 $E-13$	0.1376 $E-19$	0.2185 $E-26$

TABLE I.b

Maximum Error for (3.4) with $\lambda = 2$ as reported in [11]

N	At the nodes	At the Gauss points
2	0.627 $E-01$	0.535 $E-02$
4	0.168 $E-01$	0.815 $E-03$
8	0.434 $E-02$	0.116 $E-03$
16	0.110 $E-02$	0.155 $E-04$
32	0.278 $E-03$	0.201 $E-05$
64	0.697 $E-04$	0.255 $E-06$

Table I.c and I.d illustrate the behavior of the first four projection schemes as λ approaches the eigenvalue $\lambda = 1$.

For $\lambda = 1.001$ it is apparent that $\max|\bar{y}(x)|$ is approximately $1.7E+03$. Thus Table I.c indicates that a 1% error is assured by the first, second, third, and fourth order schemes with 63, 5, 2, and 2 nodes, respectively. But in comparison, for $\lambda = 1.00001$, $\max|\bar{y}(x)|$ is approximately $1.7E+05$ and Table I.d indicates that 1,024 nodes are required to assure a 1% error using the first order scheme. The second, third, and fourth require 16, 4, and 2 nodes respectively.

From the general error estimates (2.7) it is evident that the gain in accuracy achieved by using more complex interpolants is significantly amplified by each of the higher order schemes. For example, the preceding analysis of (3.4) employing a piecewise linear interpolant resulted in a nodal error of the order $(1/N)^{2n}$, where $N(\to \infty)$ is the number of nodes and n is the order of the scheme. By comparison a similar analysis of (3.4) based on a piecewise smooth cubic interpolant results in a nodal error of the order $(1/N)^{4n}$. The exact error expression for this case with $\lambda = 2$ is tabulated in Table I.e. Hence for the model problem (3.4) a first (second) order cubic scheme is comparable with a second (fourth) order linear scheme.

TABLE I.c

Maximum Error Associated with (3.11) for $\lambda = 1.001$

N	1st order	2nd order	3rd order	4th order
2	0.1642 $E+04$	0.1417 $E+04$	0.1618 $E+02$	0.3435 $E+00$
4	0.1444 $E+04$	0.4852 $E+02$	0.2465 $E+00$	0.1288 $E-02$
8	0.9735 $E+03$	0.2934 $E+01$	0.3819 $E-02$	0.4981 $E-05$
16	0.4228 $E+03$	0.1827 $E+00$	0.5955 $E-04$	0.1941 $E-07$
32	0.1296 $E+03$	0.1142 $E-01$	0.9300 $E-06$	0.7576 $E-10$
64	0.3433 $E+02$	0.7134 $E-03$	0.1453 $E-07$	0.2959 $E-12$
128	0.8713 $E+01$	0.4459 $E-04$	0.2270 $E-09$	0.1156 $E-14$
256	0.2186 $E+01$	0.2787 $E-05$	0.3547 $E-11$	0.4515 $E-17$
512	0.5471 $E+00$	0.1741 $E-06$	0.5541 $E-13$	0.1763 $E-19$
1024	0.1367 $E+00$	0.1087 $E-07$	0.8636 $E-15$	0.6864 $E-22$

TABLE I.d

Maximum Error Associated with (3.11) for $\lambda = 1.00001$

N	1st order	2nd order	3rd order	4th order
2	0.1718 $E+06$	0.1758 $E+06$	0.8358 $E+05$	0.3487 $E+04$
4	0.1715 $E+06$	0.2707 $E+06$	0.2422 $E+04$	0.1282 $E+02$
8	0.1705 $E+06$	0.3519 $E+05$	0.3804 $E+02$	0.4957 $E-01$
16	0.1667 $E+06$	0.1842 $E+04$	0.5933 $E+00$	0.1932 $E-03$
32	0.1530 $E+06$	0.1139 $E+03$	0.9265 $E-02$	0.7540 $E-06$
64	0.1152 $E+06$	0.7115 $E+01$	0.1448 $E-03$	0.2945 $E-08$
128	0.5794 $E+05$	0.4447 $E+00$	0.2262 $E-05$	0.1150 $E-10$
256	0.1939 $E+05$	0.2779 $E-01$	0.3534 $E-07$	0.4493 $E-13$
512	0.5295 $E+04$	0.1737 $E-02$	0.5521 $E-09$	0.1755 $E-15$
1024	0.1354 $E+04$	0.1084 $E-03$	0.8604 $E-11$	0.6831 $E-18$

TABLE I.e

Maximum Nodal Error Associated with the First Four Projection Schemes Employing a Piecewise Smooth Cubic Interpolant to Analyze (3.4) with $\lambda = 2$

N	1st order	2nd order	3rd order	4th order
2	0.6010 $E-03$	0.1051 $E-06$	0.1837 $E-10$	0.3212 $E-14$
4	0.3736 $E-04$	0.4061 $E-09$	0.4414 $E-14$	0.4798 $E-19$
8	0.2332 $E-05$	0.1582 $E-11$	0.1073 $E-17$	0.7282 $E-24$
16	01457 $E-06$	0.6175 $E-14$	0.2618 $E-21$	0.1110 $E-28$
32	0.9104 $E-08$	0.2412 $E-16$	0.6389 $E-25$	0.1693 $E-33$

Note. The mesh is uniform with N nodes.

3.3. *A Singular Integral Equation*

The weakly singular linear integral equation

$$y(x) = g(x) + \lambda \int_{-1}^{1} |x-t|^{-\alpha} y(t)\, dt, \qquad 0 < \alpha < 1, \tag{3.12}$$

appears in the theory of intrinsic viscosity, [12]. Equation (3.12) with $g(x) = x^2$ and $\alpha = 1/2$ has been the subject of many investigations. A Chebyshev series approximation is obtained in [13]. A polynomial collocation technique is discussed in [3].

The present analysis of (3.12) employs the method of Section (2.1) with

$$Py = \sum_{i=1}^{N} y(x_i^*)\, e_i(x), \qquad N \geqslant 1. \tag{3.13}$$

The projection (3.13) represents a piecewise constant interpolant of y on a uniform mesh: $-1 = x_0 < x_1 \cdots < x_N = 1$, $x_i = -1 + 2iN^{-1}$. The midpoint and the characteristic function of the interval $(x_{i-1}, x_i]$ are denoted by x_i^* and $e_i(x) = 1$ (0) for $x_{i-1} < x \leqslant x_i$ (otherwise). Because of their simplicity, such interpolants are widely used in practical computations, particularly in higher-dimensional spaces. The construction of a step function approximation to the solution of (3.12) is described in [14].

The disadvantage of (3.13) is the relatively slow rate of convergence that is associated with the accompanying Galerkin method. For instance, the estimate $\mathcal{O}(|\lambda|\,(2/N)^{1/2})$ for the convergence rate of the first order scheme (2.9) follows from (3.2) and (3.3) with $\delta = 2N^{-1}$. The corresponding estimates for the second and third order schemes are $\mathcal{O}(2\lambda^2 N^{-1})$ and $\mathcal{O}(|\lambda|^3\,(2/N)^{3/2})$, respectively. These estimates are likely to be conservative away from the end points $x = \pm 1$. They can be expected to be less conservative near $x = \pm 1$ where the solution of (3.12) fails to have a derivative. Hence this example will illustrate the use of a second and third order scheme to accelerate the nodal convergence of a first order scheme based on an elementary but practical projection. Moreover it serves to illustrate the accelerated convergence of the approximate solutions (2.14), particularly at the end points $x = \pm 1$ where the effect of the singular kernel is most pronounced. For this reason a comparison is made between the first three higher order schemes with 20 nodes and 60 nodes.

Equation (2.9) defines the first order approximation to (3.13):

$$u_1(x) = \sum_{i=1}^{N} a_i^{(1)} e_i(x). \tag{3.14}$$

The coefficients in (3.14) from the components of an N-dimensional column vector $X^{(1)}$ that satisfies

$$X^{(1)} - \lambda A X^{(1)} = E^{(1)} \tag{3.15}$$

with $E^{(1)} = Y^{(1)} + \lambda(Y^{(2)} - AY^{(1)})$. Here A is an $N \times N$ matrix with elements $\alpha_{ki} = \alpha_i(x_k^*)$, where

$$\alpha_i(x) = \int_{x_{i-1}}^{x_i} |x - t|^{-\alpha}\, dt. \tag{3.16}$$

$Y^{(1)}$ and $Y^{(2)}$ are N-dimensional column vectors with components $y_k^{(1)} = g(x_k^*)$, $y_k^{(2)} = g^{(2)}(x_k^*)$, respectively, and

$$g^{(2)}(x) = \int_{-1}^{1} |x - t|^{-\alpha}\, g(t)\, dt. \tag{3.17}$$

The first order approximation to the solution of (3.12)

$$y_1(x) = g(x) + \sum_{i=1}^{N} (a_i^{(1)} - y_i^{(1)})\, e_i(x) \tag{3.18}$$

is obtained from (2.14) with $n = 1$.

Equation (2.10) defines the second order approximation to (3.13):

$$u_2(x) = \sum_{i=1}^{N} a_i^{(2)} e_i(x). \tag{3.19}$$

The coefficients in (3.19) form the components of the solution vector to

$$X^{(2)} - \lambda A X^{(2)} - \lambda^2 (B - A^2)\, X^{(2)} = E^{(2)}, \tag{3.20}$$

where $E^{(2)} = E^{(1)} + \lambda^2(Y^{(3)} - AY^{(2)} - (B - A^2)\, Y^{(1)})$. The elements of the $N \times N$ matrix B are $\beta_{ki} = \beta_i(x_k^*)$ and the components of the column vector $Y^{(3)}$ are $y_k^{(3)} = g^{(3)}(x_k^*)$. Here

$$\beta_i(x) = \int_{x_{i-1}}^{x_i} \int_{-1}^{1} |x - t|^{-\alpha}\, |t - \xi|^{-\alpha}\, dt\, d\xi \tag{3.21}$$

and

$$g^{(3)}(x) = \int_{-1}^{1} \int_{-1}^{1} |x - t|^{-\alpha}\, |t - \xi|^{-\alpha}\, g(\xi)\, dt\, d\xi. \tag{3.22}$$

Equation (2.14) with $n = 2$ gives the second order approximate solution of (3.12) as

$$y_2(x) = g(x) + \lambda g^{(2)}(x) + \lambda \sum_{i=1}^{N} (a_i^{(2)} - y_i^{(1)})\, \alpha_i(x) + \sum_{i=1}^{N} \left\{ a_i^{(2)} - y_i^{(1)} - \lambda y_i^{(2)} - \lambda \sum_{k=1}^{N} \alpha_{ik}(a_k^{(2)} - y_k^{(1)}) \right\} e_i(x). \tag{3.23}$$

The third order approximation to (3.13) is obtained by solving (2.5) with $n = 3$. Thus the coefficients of

$$u_3(x) = \sum_{i=1}^{N} a_i^{(3)} e_i(x) \tag{3.24}$$

are defined by the solution vector of

$$X^{(3)} - \lambda A X^{(3)} - \lambda^2 (B - A^2) X^{(3)} - \lambda^3 (C - BA - AB + A^3) X^{(3)} = E^{(3)}, \tag{3.25}$$

where

$$E^{(3)} = E^{(2)} + \lambda^3 (Y^{(4)} - A Y^{(3)} - (B - A^2) Y^{(2)} - (C - BA - AB + A^3) Y^{(1)}),$$

C is an $N \times N$ matrix with elements $\gamma_{kl} = \gamma_l(x_k^*)$ and $Y^{(4)}$ is a column vector with components $y_k^{(4)} = g^{(4)}(x_k^*)$. Here

$$\gamma_l(x) = \int_{x_{l-1}}^{x_l} \int_{-1}^{1} \int_{-1}^{1} |x - \tau|^{-\alpha} |\tau - \xi|^{-\alpha} |\xi - \eta|^{-\alpha} \, d\xi \, d\tau \, d\eta \tag{3.26}$$

and

$$g^{(4)}(x) = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} |x - \tau|^{-\alpha} |\tau - \xi|^{-\alpha} |\xi - \eta|^{-\alpha} g(\eta) \, d\xi \, d\tau \, d\eta. \tag{3.27}$$

The third order approximate solution of (3.12), obtained from (2.14) with $n = 3$, is

$$\begin{aligned} y_3(x) = {} & g(x) + \lambda g^{(2)}(x) + \lambda^2 g^{(3)}(x) + \lambda^2 \sum_{i=1}^{N} v_i^{(1)} \beta_i(x) \\ & + \lambda \sum_{i=1}^{N} \left\{ v_i^{(2)} - \lambda \sum_{k=1}^{N} \alpha_{ik} v_k^{(1)} \right\} \alpha_i(x) \\ & + \sum_{i=1}^{N} \left\{ v_i^{(3)} - \lambda \sum_{k=1}^{N} \alpha_{ik} v_k^{(2)} + \lambda^2 \sum_{k=1}^{N} \left(\sum_{l=1}^{N} \alpha_{il} \alpha_{lk} - \beta_{ik} \right) v_k^{(1)} \right\} e_i(x), \end{aligned} \tag{3.28}$$

where $v_i^{(1)} = a_i^{(3)} - y_i^{(1)}$; $v_i^{(2)} = v_i^{(1)} - \lambda y_i^{(2)}$ and $v_i^{(3)} = v_i^{(2)} - \lambda^2 y_i^{(3)}$.

For $g(x) = x^2$ and $\alpha = \frac{1}{2}$ the integrals (3.16), (3.17), (3.21), (3.22), (3.26) and (3.27) can be evaluated and the systems (3.15), (3.20) and (3.25) analyzed without resorting to numerical integrations. The resulting approximations for selected nodal values of the solution to (3.12) with $\lambda = -\frac{1}{2}$ are tabulated in Tables II.a and II.b for 20 and 60 nodes, respectively.

Tables II.a and II.b show clearly the convergence of the nodal values with increasing order as well as increasing N. Moreover, of the three schemes the third order changes least as N increases from 20 to 60.

This reflects the greater accuracy of the 20-point third order scheme which is

TABLE II.a

Nodal Values of the Solution to (3.12) with $\lambda = -\frac{1}{2}$ Using a 20-Point Piecewise Constant Interpolant

Node	1st order	2nd order	3rd order
0.95	0.520439	0.519460	0.519568
0.85	0.367219	0.366992	0.366987
0.75	0.255111	0.254871	0.254892
0.65	0.165245	0.164981	0.165017
0.55	9.21965 $E-2$	9.19131 $E-2$	9.19584 $E-2$
0.45	3.34467 $E-2$	3.31491 $E-2$	3.32009 $E-2$
0.35	−1.24062 $E-2$	−1.27142 $E-2$	−1.26581 $E-2$
0.25	−4.62116 $E-2$	−4.65267 $E-2$	4.64677 $E-2$
0.15	−6.84947 $E-2$	−6.88144 $E-2$	−6.87536 $E-2$
0.05	−7.95645 $E-2$	−7.98864 $E-2$	−7.98248 $E-2$

TABLE II.b

Nodal Values of the Solution to (3.12) with $\lambda = -\frac{1}{2}$ Using a 60-Point Piecewise Constant Interpolant

Node	1st order	2nd order	3rd order
0.95	0.519717	0.519556	0.519558
0.85	0.367077	0.366990	0.366993
0.75	0.254961	0.254887	0.254891
0.65	0.165077	0.165008	0.165013
0.55	9.20132 $E-2$	9.19474 $E-2$	9.19525 $E-2$
0.45	3.32523 $E-2$	3.31881 $E-2$	3.31936 $E-2$
0.35	−1.26086 $E-2$	−1.26719 $E-2$	−1.26662 $E-2$
0.25	−4.64195 $E-2$	−4.64821 $E-2$	−4.64764 $E-2$
0.15	−6.87063 $E-2$	−6.87684 $E-2$	−6.87627 $E-2$
0.05	−7.97778 $E-2$	−7.98399 $E-2$	−7.98341 $E-2$

TABLE II.c

First, Second and Third Order Estimates for the Solution of (3.12) at the End Point $x = 1$

Number of nodes	1st order	2nd order	3rd order
20	0.617939	0.692628	0.688513
40	0.624227	0.691734	0.688680
60	0.630547	0.691097	0.688766
60-node error	0.058568	0.001982	0.000349

Note. These estimates, for the case $g(x) = x^2$, $\alpha = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$, are a result of a piecewise constant analysis employing (3.13). They are obtained by evaluating of (3.18), (3.23) and (3.28) at $x = 1$. The error estimates are based on a 70-node second order piecewise linear analysis of (3.12), see Table II.d.

TABLE II.d

First and Second Order Estimates for the Solution of (3.12) at the End Point $x = 1$.

Number of nodes	1st order	2nd order
20	0.680747	0.690056
40	0.684497	0.689349
60	0.685854	0.689162
60-node error	0.003261	0.000047

Note. These estimates, for the case $g(x) = x^2$, $\alpha = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$, are a result of a piecewise linear analysis employing (3.29). The error estimates are based on the 70-node second order result (0.689115).

estimated at 10^{-5}. The corresponding estimates for the maximum error associated with the 20-point second and first order schemes are 10^{-4} and 10^{-3}, respectively.

Since the functional $y(1)$ is not included in the definition (3.13) the approximate solutions (3.18), (3.23) and (3.28) must be employed to obtain first, second and third order estimates for the solution of (3.12) at the end point $x = 1$. These estimates are tabulated in Table II.c for the case $\lambda = -\frac{1}{2}$. Although each of the three orders converges more slowly than in the interior, the accelerated convergence of the second and third order schemes relative to the first order is clearly evident.

As in the Model Problem (Section 3.2) further acceleration of the convergence rate can be achieved by employing either a fourth order scheme based on (3.13) or a lower order scheme based on an alternate projection. This is illustrated in Table II.d where the end point estimates of $y(1)$, obtained from a first and second order analysis of (3.12) employing

$$Py = \sum_{i=1}^{N} y(x_i)\, r_i(x), \tag{3.29}$$

are tabulated. Here (3.29) represents a piecewise linear interpolant defined on the uniform mesh $-1 = x_0 < x_1 < \cdots < x_N = 1$, $x_i = -1 + 2iN^{-1}$, with the r_i given by (3.5)–(3.7). Comparison of Tables II.c and II.d indicate that for (3.12) with $\alpha = \frac{1}{2}$ a first (second) order piecewise linear scheme is comparable to a second (fourth) order piecewise constant scheme.

3.4. *Love's Equation*

Numerical investigations of the Love equations [15],

$$y(x) \pm \pi^{-1} \int_{-1}^{1} \left(\frac{d}{d^2 + (x-t)^2} \right) y(t)\, dt = 1, \qquad |x| \leqslant 1, \tag{3.30}$$

have been reported by several authors. Finite-difference [16], and quadrature methods, [17], have been applied to (3.30) with $d = 1$. The use of polynomial collocation and Chebyshev series techniques for $d > 0$ are discussed in [2] and [13],

respectively. The present analysis of (3.30) employs the first and second order schemes associated with the piecewise constant interpolant (3.13). Also reported, for comparative purposes, are the results of an analysis based on the piecewise linear interpolant (3.29).

The first order schemes

$$a_k^{\pm} \pm \pi^{-1} \sum_{i=1}^{N} \alpha_{ki} a_i^{\pm} = 1, \qquad 1 \leqslant k \leqslant N \tag{3.31}$$

with

$$\alpha_{ki} = \tan^{-1}\left(\frac{x_k^* - x_{i-1}}{d}\right) - \tan^{-1}\left(\frac{x_k^* - x_i}{d}\right)$$

determine the coefficients of a step function, $u_1^{\pm}(x) = \sum_{i=1}^{N} a_i^{\pm} e_i(x)$, approximation to

$$Py^{\pm}(x) = \sum_{i=1}^{N} y^{\pm}(x_i^*)\, e_i(x).$$

Here $y^{\pm}$ are the solutions of (3.30). And x_i^*, $e_i(x) = 1$ (0) for $x_{i-1} < x \leqslant x_i$ (otherwise) are the midpoint and characteristic function of the interval $(x_{i-1}, x_i]$ defined by the nodes $x_i = -1 + 2iN^{-1}$, $0 \leqslant i \leqslant N$. In addition it follows from (2.14) with $n = 1$ that the step functions $y_1^{\pm}(x) = u_1^{\pm}(x)$ are first order interpolants to the solutions of (3.30).

The second order schemes

$$b_k^{\pm} \pm \pi^{-1} \sum_{i=1}^{N} \alpha_{ki} b_i^{\pm} - \pi^{-2} \sum_{i=1}^{N} \left(\beta_{ki} - \sum_{j=1}^{N} \alpha_{kj}\alpha_{ji}\right) b_i^{\pm} = 1, \qquad 1 \leqslant k \leqslant N, \tag{3.32}$$

where

$$\beta_{ki} = \int_{-1}^{1} \left(\frac{d}{d^2 + (x_k^* - \xi)^2}\right)\left(\tan^{-1}\left(\frac{x_i - \xi}{d}\right) - \tan^{-1}\left(\frac{x_{i-1} - \xi}{d}\right)\right) d\xi \tag{3.33}$$

determine the coefficients of $u_2^{\pm}(x) = \sum_{i=1}^{N} b_i^{\pm} e_i(x)$. The approximate solutions

$$y_2^{\pm}(x) = \sum_{k=1}^{N} \left\{ \left(b_k^{\pm} \pm \pi^{-1} \sum_{i=1}^{N} \alpha_{ki} b_i^{\pm}\right) e_k(x) \right.$$
$$\left. \pm \pi^{-1} b_k^{\pm} \left(\tan^{-1}\left(\frac{x - x_k}{d}\right) - \tan^{-1}\left(\frac{x - x_{k-1}}{d}\right)\right)\right\} \tag{3.34}$$

obtained from (2.14) with $n = 2$ represent second order interpolants to the solutions of (3.30).

TABLE III.a

Nodal Values of the Solutions to (3.30) with $d = 1$ Using a 5- and 15-Point Piecewise Constant Interpolant

	The nodal values of $y^+(x)$				The nodal values of $y^-(x)$			
	First order		Second order		First order		Second order	
Node	$N=5$	$N=15$	$N=5$	$N=15$	$N=5$	$N=15$	$N=5$	$N=15$
$0.9\bar{3}$	—	0.744301	—	0.744258	—	1.67103	—	1.67083
0.8	0.722834	0.722526	0.722488	0.722488	1.73246	1.73093	1.73075	1.73075
$0.6\bar{6}$	—	0.703099	—	0.703068	—	1.78542	—	1.78525
$0.5\bar{3}$	—	0.686751	—	0.686729	—	1.83223	—	1.83208
0.4	0.674028	0.673903	0.673888	0.673888	1.87087	1.86977	1.86964	1.86964
$0.2\bar{6}$	—	0.664725	—	0.664716	—	1.89707	—	1.89695
$0.1\bar{3}$	—	0.659239	—	0.659233	—	1.91361	—	1.91350
0.0	0.657457	0.657416	0.657412	0.657412	1.92003	1.91914	1.91903	1.91903

The solutions of (3.31) and (3.32) with $d = 1$ are listed in Table III.a for $N = 5$ and 15. A Romberg integration technique was used to evaluate (3.33). Inspection of Table III.a indicates that the 5-point second order scheme is at least 100 times more accurate than the corresponding first order scheme and yields approximations to the nodal values of y^+ and y^- with errors of the order 10^{-6}.

The second order interpolants (3.34) are tabulated in Table III.b for selected arguments. The results obtained in [2] using a 20-point polynomial collocation scheme are listed in Table III.c. Comparison of Tables III.b and III.c indicate that the

TABLE III.b

Selected Values of the Second Order Interpolant (3.34) to the Solutions of (3.30) with $d = 1$ Using 5, 7 and 10 nodes

	Interpolated values of $y^+(x)$			Interpolated values of $y^-(x)$		
x	$N=5$	$N=7$	$N=10$	$N=5$	$N=7$	$N=10$
1.0	0.755783	0.755739	0.755725	1.63988	1.63976	1.63971
0.95	0.747150	0.747112	0.747100	1.66325	1.66314	1.66310
0.90	0.738685	0.738654	0.738646*	1.68628	1.68619	1.68617*
0.80	0.722488*	0.722476	0.722500	1.73075*	1.73071	1.73078
0.60	0.694579	0.694488	0.694493	1.81002	1.80976	1.80978
0.40	0.673888*	0.673911	0.673898	1.86964*	1.86970	1.86967
0.20	0.661540	0.661502	0.661520	1.90666	1.90655	1.90660
0.0	0.657412*	0.657412*	0.657410	1.91903*	1.91903*	1.91903

Note. The accuracy of the interpolant (3.34) varies with x as well as N. For a fixed N the greatest accuracy occurs at the nodes, indicated by an asterisk, where the error is approximately 10^{-6}.

TABLE III.c

20-Node Polynomial Collocation Solution of (3.30) with $d = 1$ as Reported in [2]

x	$y^+(x)$	$y^-(x)$
1.0	0.7557	1.640
0.95	0.7471	1.663
0.90	0.7386	1.686
0.80	0.7225	1.731
0.60	0.6945	1.810
0.40	0.6739	1.870
0.20	0.6615	1.907
0.0	0.6574	1.919

second order interpolants (3.34) employing seven nodes are as accurate as a polynomial scheme employing approximately three times as many nodes.

An easily obtained estimate, $\mathcal{O}(N^{-2})$, for the accuracy of (3.34) implies that these second order interpolants are comparable with first order interpolants derived from (2.14) using the piecewise linear projection (3.29). Acceleration of these schemes results in second order interpolants with estimated accuracy $\mathcal{O}(N^{-4})$. Selected values of these interpolants are tabulated in Table III.d.

TABLE III.d

Selected Values of Second Order Approximate Solutions of (3.30) Obtained from (2.14) with $n = 2$ and Employing the Piecewise Linear Interpolant (3.29)

	Interpolated values of $y^+(x)$			Interpolated values of $y^-(x)$		
x	3 nodes	4 nodes	5 nodes	3 nodes	4 nodes	5 nodes
1.0	0.755814	0.755734	0.755723	1.63904	1.63958	1.63966
0.95	0.747105	0.747096	0.747097	1.66218	1.66293	1.66304
0.90	0.73857	0.738630	0.738642	1.68501	1.68596	1.68611
0.80	0.722256	0.722445	0.722477	1.72912	1.73046	1.73067
0.60	0.694034	0.694418	0.694473	1.80744	1.80938	1.80966
0.40	0.673424	0.673876	0.673868	1.86720	1.86938	1.86951
0.20	0.661252	0.661449	0.661480	1.90461	1.90615	1.90641
0.0	0.657550	0.657294	0.657419	1.91808	1.91843	1.91898

Note. Comparison with the results tabulated in Table III.c indicate that the five node approximation is as accurate as a polynomial scheme employing four times as many nodes.

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